Cylindrical Wave Approach (CWA) for the electromagnetic modelling of 2D GPR scenarios: lossy media

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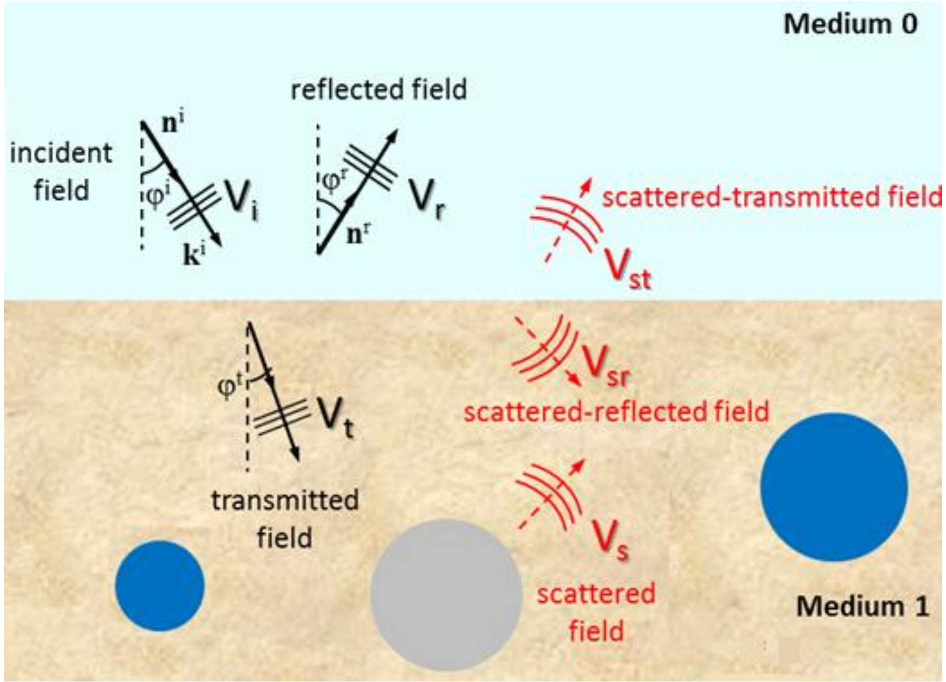
Outline

- The Cylindrical Wave Approach (CWA): an introduction
- Lossy media: convergence problems
- Analytical solutions
- Conclusions

Scattering by buried cylindrical objects

Flat interface between lossless media

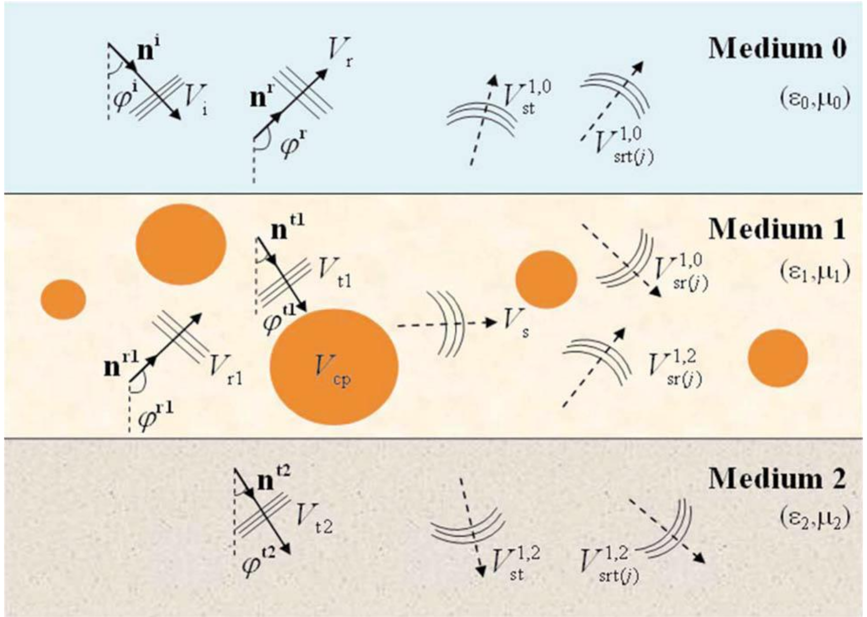
1) Homogeneous, isotropic, lossless media
2) Dielectric cylinders with different radii
3) Arbitrarily large number of cylinders
4) High computational precision
5) Fast evaluation



Scattering by buried cylindrical objects (2)

Stratified media

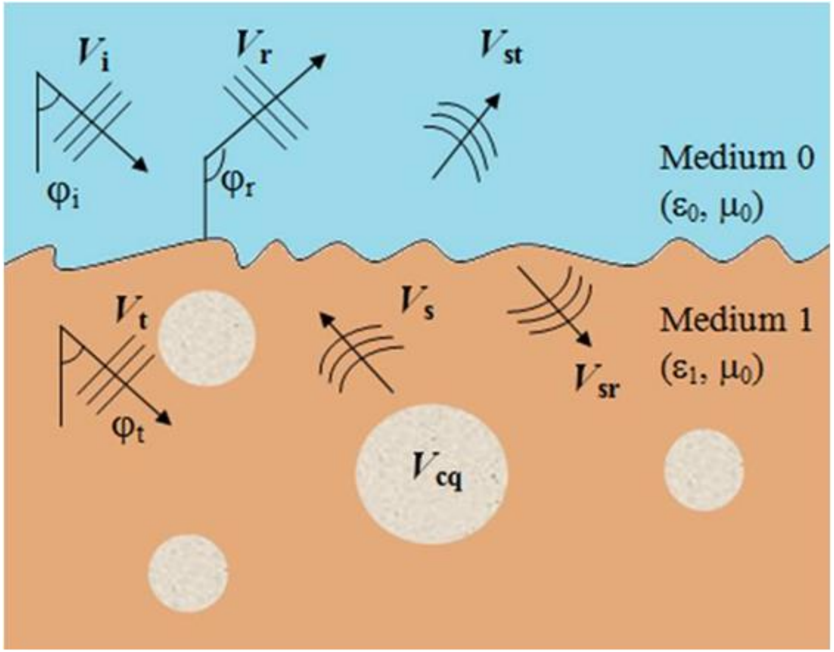
1) Homogeneous, isotropic, lossless media
2) Dielectric cylinders with different radii
3) Arbitrarily large number of cylinders
4) High computational precision
5) Fast evaluation
6) Arbitrary stratification



Scattering by buried cylindrical objects (3)

Rough Surfaces

1) Homogeneous, isotropic, lossless media
2) Dielectric cylinders with different radii
3) Arbitrarily large number of cylinders
4) High computational precision
5) Fast evaluation
6) Arbitrary stratification
7) Arbitrary shape of the interface



Scattering by buried cylindrical objects (4)

LOSSY MEDIA?

Cylindrical wave expansions

Transmitted wave:

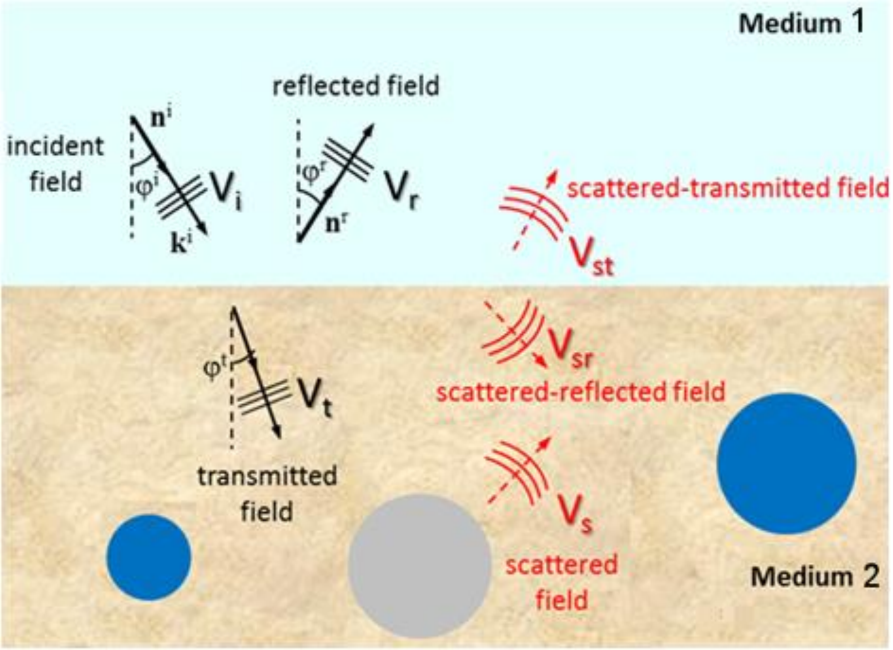
$$V_t(x,y) = V_0 e^{-j\left(k_x x + k_y y\right)}$$

$$\Downarrow$$

$$V_t(\rho,\theta) = V_0 e^{-jk\rho(\cos\theta\cos\varphi + \sin\theta\sin\varphi)} = V_0 e^{-jk\rho\cos(\theta-\varphi)}$$

$$\Downarrow$$

$$V_t(\rho,\theta) = V_0 \sum_{m=-\infty}^{+\infty} j^{-m} e^{jm\varphi} J_m(k_2\rho)\, e^{-jm\theta}$$



Cylindrical wave expansions

Transmitted wave:

We can work with the scalar potential because the problem is 2D

$$V_t(x,y) = V_0 e^{-j(k_x x + k_y y)}$$

$$\Downarrow$$

$$V_t(\rho,\theta) = V_0 e^{-jk\rho(\cos\theta\cos\varphi + \sin\theta\sin\varphi)} = V_0 e^{-jk\rho\cos(\theta-\varphi)}$$

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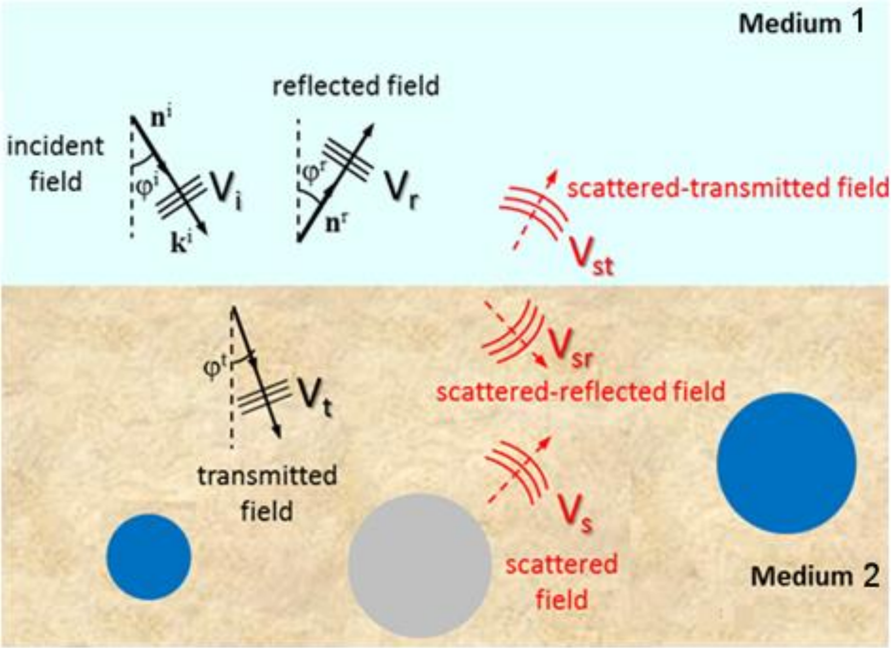
$$V_t(\rho,\theta) = V_0 \sum_{m=-\infty}^{+\infty} j^{-m} e^{jm\varphi} J_m(k_2\rho)\, e^{-jm\theta}$$



The scattered wave can be expanded in cylindrical waves as well

Cylindrical wave expansions (2)

Scattered wave:

$$V_s(\rho,\theta) = V_0 \sum_{m=-\infty}^{+\infty} j^{-m} e^{jm\varphi} c_m H_m^{(2)}(k_2\rho)\, e^{-jm\theta}$$

The problem reduces to the determination of the coefficients

Scattered-Reflected wave:

$$V_{sr} = \sum_{m=-\infty}^{+\infty} c_m j^{-m} RW_m\left[-\mathrm{n}_2(\xi+2\chi), \mathrm{n}_2\zeta\right] \mathrm{e}^{jm\varphi}$$

Scattered-Transmitted wave:

$$V_{st} = \sum_{m=-\infty}^{+\infty} c_m j^{-m} TW_m(\xi,\zeta,\chi)\, \mathrm{e}^{jm\varphi}$$

With:

$$\xi = k_0 x$$
$$\zeta = k_0 y$$
$$\chi = k_0 h$$
$$k_y = k_0 n_y$$

and

$$RW_m(\xi,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{21}(n_y) F_m(\xi,n_y) \exp(-jn_y\zeta)\, \mathrm{d}n_y$$

$$TW_m(\xi,\zeta,\chi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} T_{21}(n_y) F_m(\xi,n_y) \exp(-jn_y\zeta)\, \mathrm{d}n_y$$

Cylindrical wave expansions (2)

Scattered wave:

$$V_s(\rho,\theta) = V_0 \sum_{m=-\infty}^{+\infty} j^{-m} e^{jm\varphi} c_m H_m^{(2)}(k_2\rho)\, e^{-jm\theta}$$

The problem reduces to the determination of the coefficients

Scattered-Reflected wave:

$$V_{sr} = \sum_{m=-\infty}^{+\infty} c_m j^{-m} RW_m\left[-n_2(\xi+2\chi), \mathrm{n}_2\,\zeta\right]\mathrm{e}^{jm\varphi}$$

Scattered-Transmitted wave:

$$V_{st} = \sum_{m=-\infty}^{+\infty} c_m j^{-m} TW_m(\xi,\zeta,\chi)\,\mathrm{e}^{jm\varphi}$$

With:

$$\xi = k_0 x$$
$$\zeta = k_0 y$$
$$\chi = k_0 h$$
$$k_y = k_0 n_y$$

and

$$\boxed{\begin{aligned} RW_m(\xi,\zeta) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{21}(n_y) F_m(\xi,n_y)\exp(-jn_y\zeta)\,\mathrm{d}n_y \\ TW_m(\xi,\zeta,\chi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} T_{21}(n_y) F_m(\xi,n_y)\exp(-jn_y\zeta)\,\mathrm{d}n_y \end{aligned}}$$

Cylindrical wave expansions (3)

Boundary conditions for the determination of the unknown coefficients:

$$[V_t + V_s + V_{sr}]_{\rho=a} = 0 \qquad \text{for E-polarization}$$

$$\frac{\partial}{\partial\rho}[V_t + V_s + V_{sr}]_{\rho=a} = 0 \qquad \text{for H-polarization}$$

Thanks to the orthogonality of the imaginary exponential,
these conditions become an infinite set of linear equations

By an efficient truncation criterion, it is possible to find a finite number of coefficients
to accurately describe the scattered field

At this point the analytical formulation is complete,
and the solution of the fields becomes a purely numerical problem

Cylindrical wave expansions (4)

The main numerical effort is in the computation of the reflected and transmitted waves:

$$RW_m(\xi,\zeta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{21}(n_y)F_m(\xi,n_y)\exp(-jn_y\zeta)\,\mathrm{d}n_y$$

$$TW_m(\xi,\zeta,\chi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}T_{21}(n_y)F_m(\xi,n_y)\exp(-jn_y\zeta)\,\mathrm{d}n_y$$

The kernel of the integrals presents an highly oscillating behavior

Considering the cylindrical wave spectrum:

$$F_m(\xi,n_y)=\frac{2\exp\left[-j|\xi|\sqrt{n_2^2-n_y^2}\pm jm\arccos(n_y/n_2)\right]}{\sqrt{n_2^2-n_y^2}}$$

The imaginary exponential, while its exponent is purely imaginary, varies very fast with n_y

Special quadrature algorithm

To efficiently compute the integrals, a special quadrature algorithm has been proposed

$$F_m(\xi, n_y) = \frac{2\exp\left[-j|\xi|\sqrt{n_2^2 - n_y^2} \pm jm\arccos(n_y / n_2)\right]}{\sqrt{n_2^2 - n_y^2}}$$

In the lossless case, by a change of variable in the integral, i.e, $n_y / n_2 = n'_y$

$$F_m(\xi', n'_y) = \frac{2\exp\left[-j|\xi'|\sqrt{1 - n'^2_y} \pm jm\arccos(n'_y)\right]}{\sqrt{1 - n'^2_y}}$$

Calling $n'_y = \cos t$, in the interval $|n'_y| < 1$, where the kernel presents the fastest oscillations

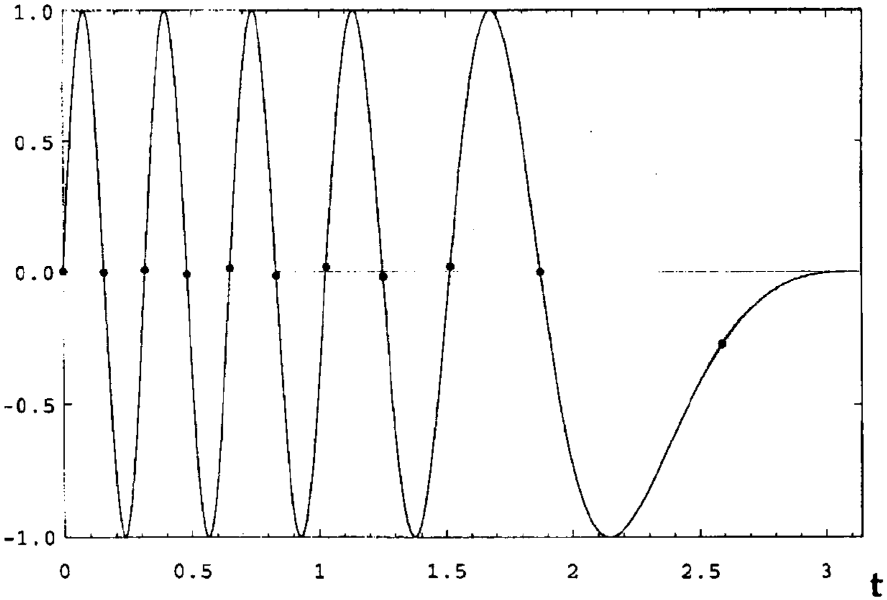
$$F_m(\xi', t) = \frac{2\exp\left[-j|\xi'|\sin t \pm jmt\right]}{\sin t}$$

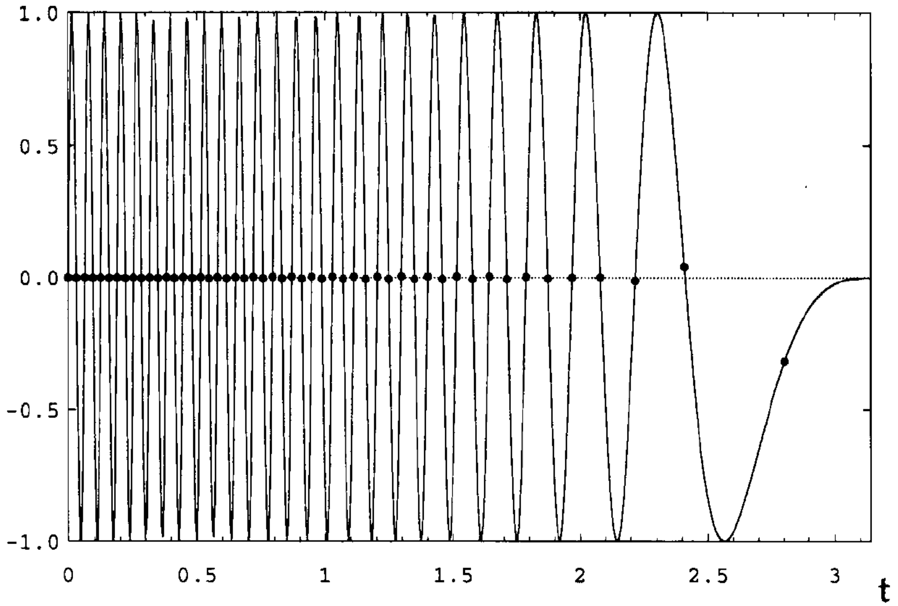
The oscillating term becomes harmonic, and its local frequency can be analytically determined

Special quadrature algorithm (2)

$$F_m\left(\xi',t\right)=\frac{2\exp\left[-j\left|\xi'\right|\sin t\pm\ jmt\right]}{\sin t}$$

Knowing the local frequency, it is possible to build a quadrature algorithm where the function is divided in non-oscillating intervals:





Lossy media

The good news is that in the lossy media
the cylindrical wave approach works in the same way!

The bad news is that in the lossy media
the numerical computation of the integrals diverges!

Why?

Lossy media

The good news is that in the lossy media
the cylindrical wave approach works in the same way!

The bad news is that in the lossy media
the numerical computation of the integrals diverges!

Why?

From an analytical point of view the functions and the integrals
can be extended in the complex plane without any problem

Analytically the integrals should converge, as well as in the lossless case,
but when they are computed with the numerical algorithm, they diverge

A convergence problem

The solution to the problem was trivial, but captious!

It starts from the change of variables inside the integrals, and in particular in $n_y / n_2 = n'_y$

In the lossless case, n_2 is a real quantity, then the change of variable is innocent!

In the lossy case, n_2 is a complex quantity, changing the convergence domain of the integral.

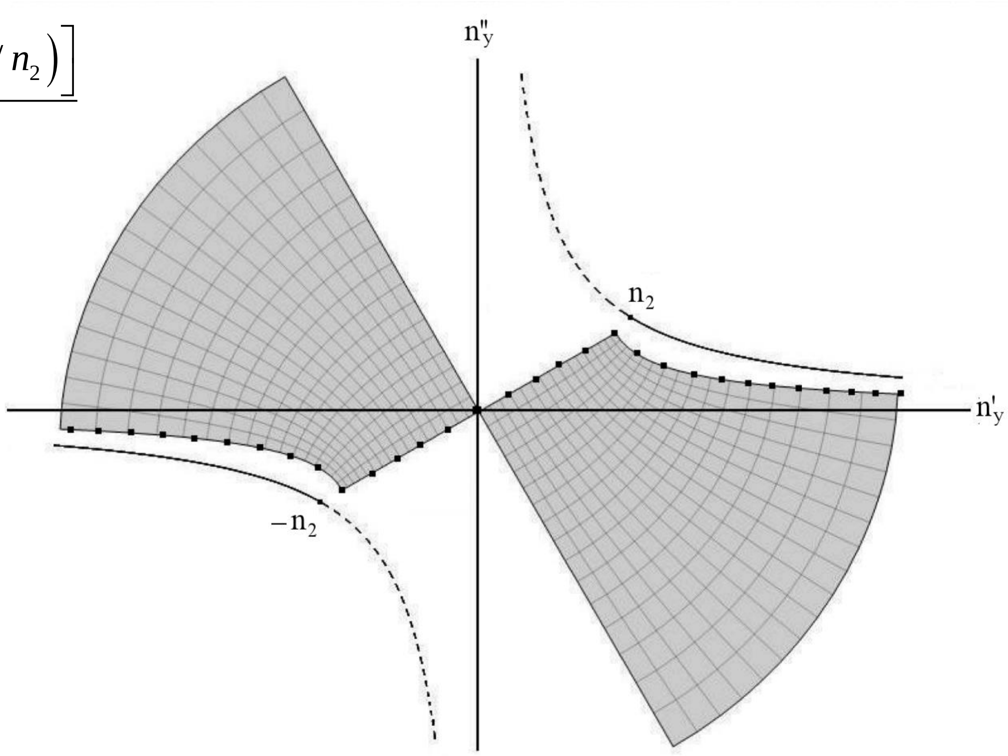
Considering the spectrum of the cylindrical wave: $F_m\left(\xi, n_y\right) = \dfrac{2\exp\left[-j\left|\xi\right|\sqrt{n_2^2 - n_y^2} \pm jm\arccos\left(n_y / n_2\right)\right]}{\sqrt{n_2^2 - n_y^2}}$

We can compute its convergence domain on the complex plane!

Convergence domain

$$F_m\left(\xi,n_y\right)=\frac{2\exp\left[-j\left|\xi\right|\sqrt{n_2^2-n_y^2}\pm jm\arccos\left(n_y/n_2\right)\right]}{\sqrt{n_2^2-n_y^2}}$$

The real axis is entirely contained in the convergence domain



When the change of variable is implemented, the convergence domain is clockwise rotated, and the real axis is not anymore contained

Convergence solution

The conclusion is that in the lossy case we cannot implement the change of variable

We must integrate the function in its original form:

$$F_m(\xi, n_y) = \frac{2\exp\left[-j|\xi|\sqrt{n_2^2 - n_y^2} \pm jm\,\mathrm{arccos}(n_y / n_2)\right]}{\sqrt{n_2^2 - n_y^2}}$$

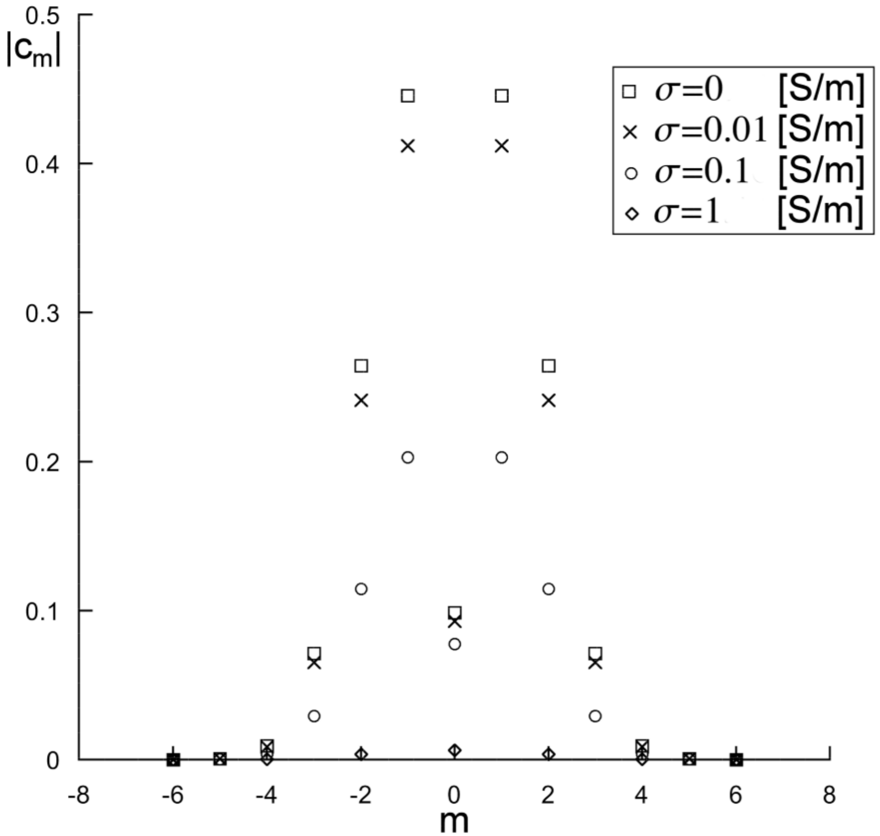
The local frequency cannot be computed analytically!

We built a numerical code to find the zeros of the exponent, in order to numerically compute the local frequency of the exponential function

The CWA started to work in lossy media.
However, the code is slower than the one in the lossless case.

Convergence solution (2)

Convergence of the scattering coefficients vs conductivity



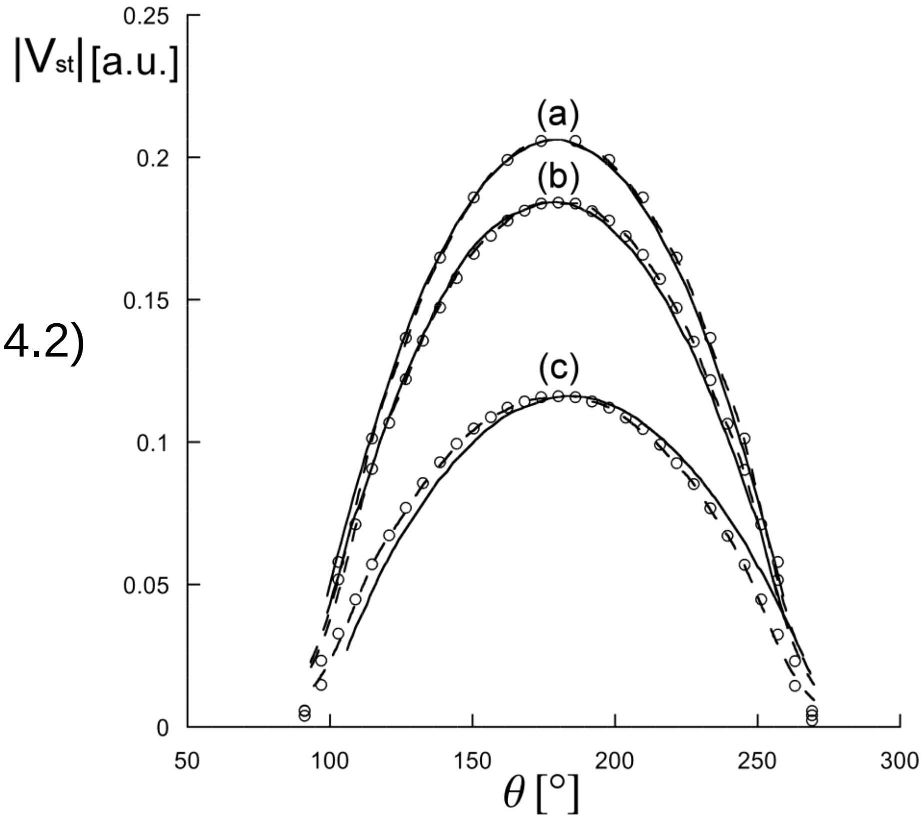
Comparisons

Comparisons with the literature and with a Finite Element Method (FEM)

Solid line: literature

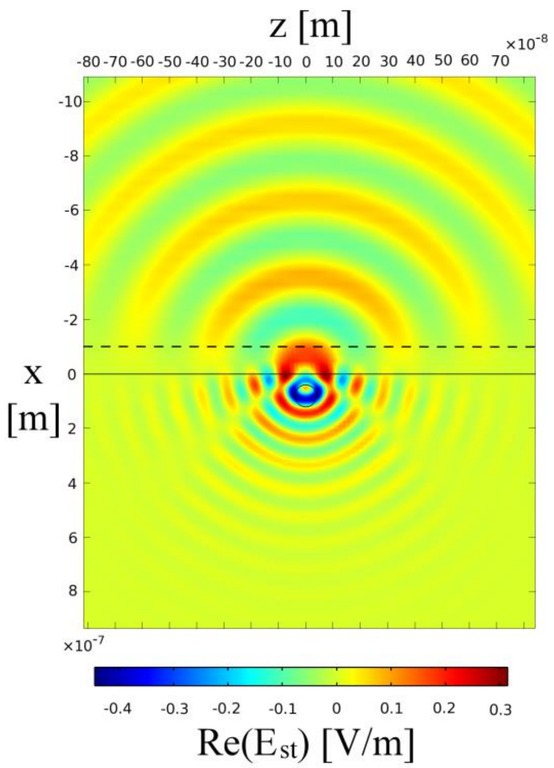
Dashed line: FEM (Comsol Multiphysics 4.2)

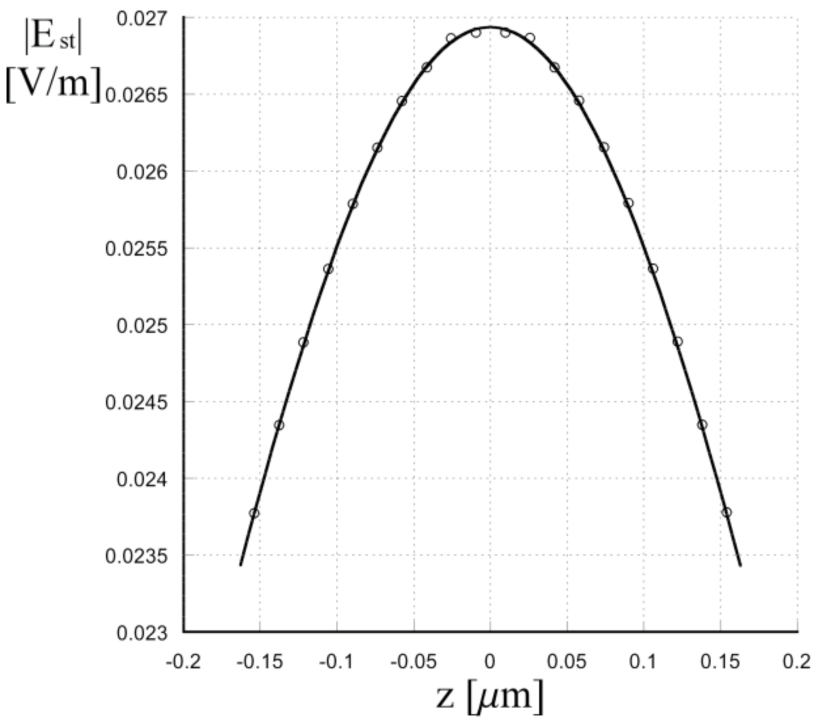
Circles: numerical code



Comparisons (2)

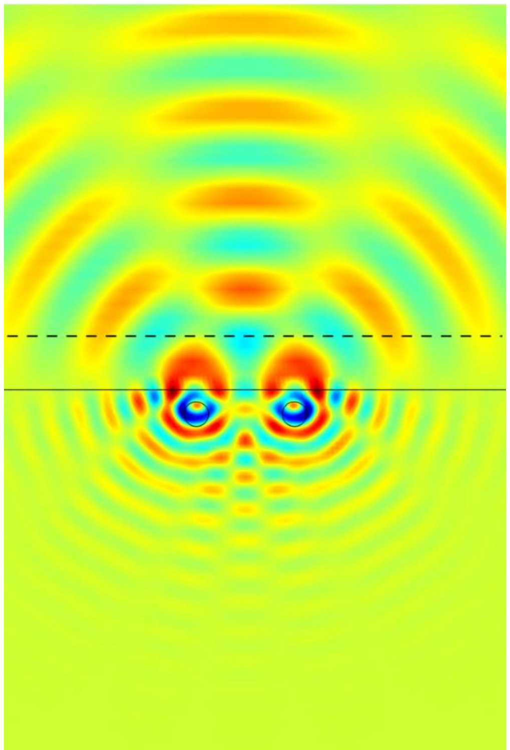
Comparisons with a Finite Element Method (FEM)



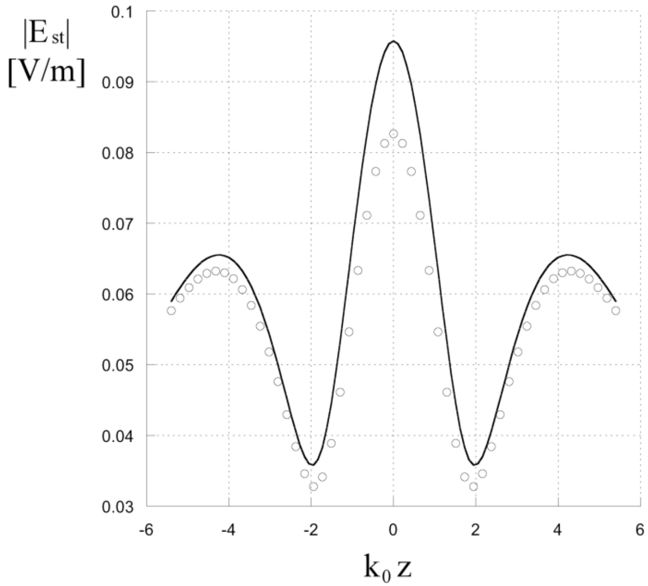


Solid line: FEM Circles: numerical code

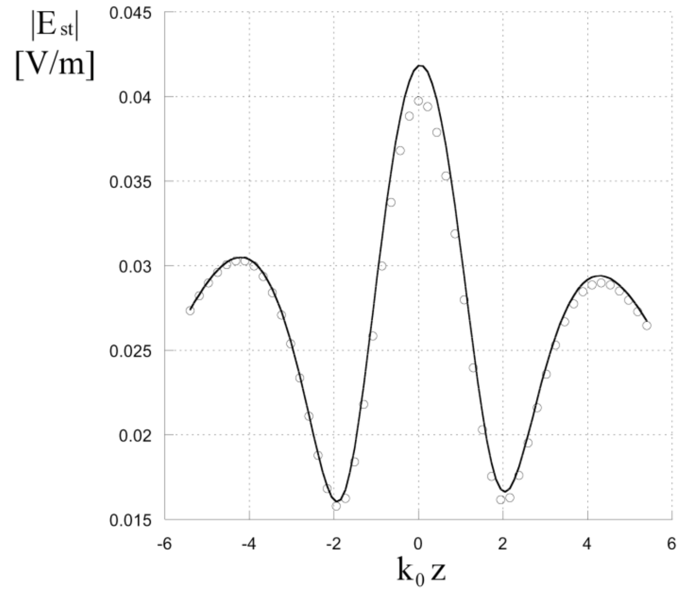
Cylinders interactions



Lossless case



Lossy case

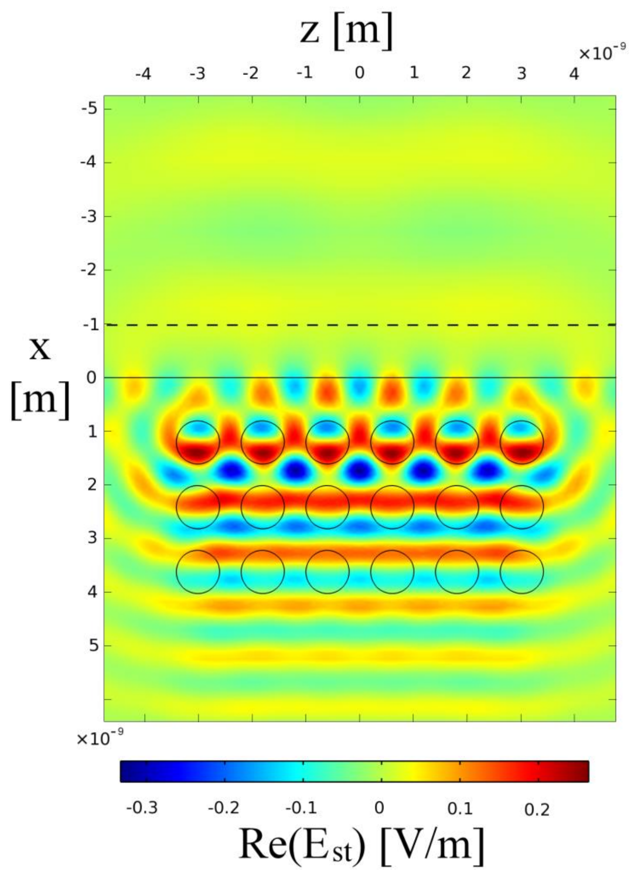


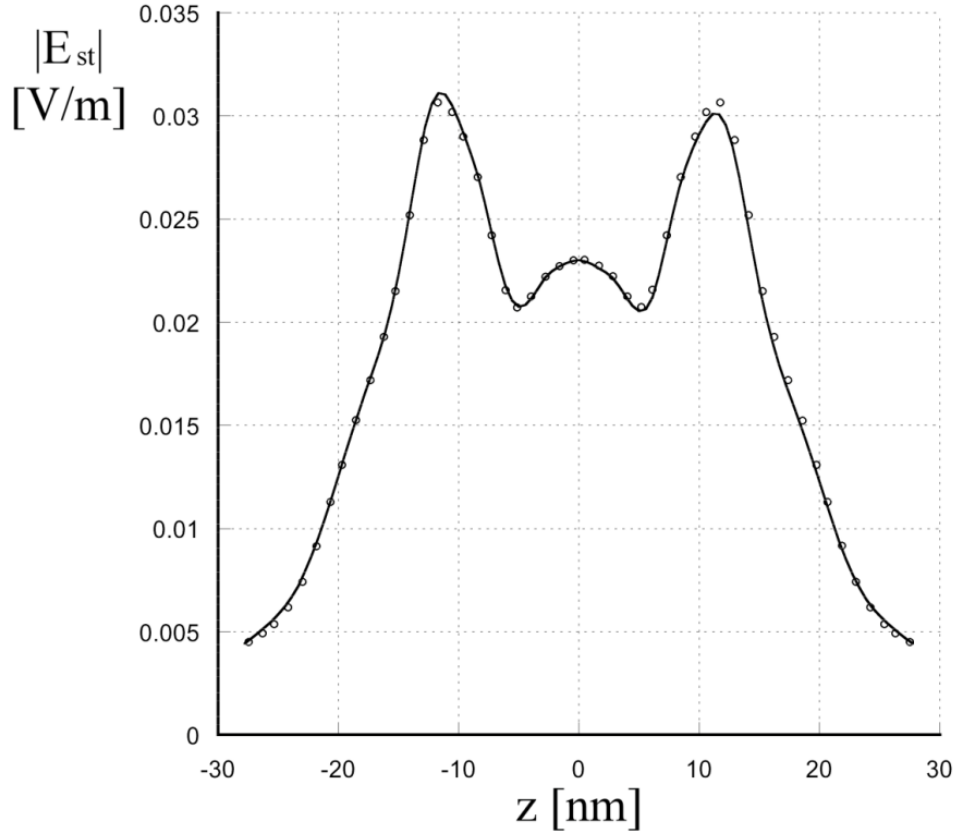
Solid line: with interactions

Circles: without interactions

a = 0.04 µm, h = 0.08 µm, d = 0.16 µm, ε_2 = 8.7 - j1.2 (SiC) and ε_c = 1.44 - j1.3 (Ag) at λ_0 = 0.3 µm

Cylinders array





Solid line: FEM   Circles: numerical code

a = 0.04 μm, h = 0.08 μm, ε_2 = 8.7 - j1.2 (SiC) and ε_c = 1.44 - j1.3 (Ag) at λ_0 = 0.3 μm

Alternative approach

The solution found presents several limitations:

1. The accuracy decreases when the losses increases
2. The accuracy decreases when the number of cylinders increases
3. Higher is the conductivity, slower is the code

Alternative approach?

Let us come back again to the integral:

$$RW_m(\xi,\zeta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{21}(n_y)F_m(\xi,n_y)\exp(-jn_y\zeta)\,\mathrm{d}n_y$$

Are we able to compute it analytically?

Integral kernel

$$RW_m\left(\xi,\zeta\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{21}\left(n_y\right)F_m\left(\xi,n_y\right)\exp\left(-jn_y\zeta\right)\mathrm{d}n_y$$

If we look inside the integral kernel:

$$F_m\left(\xi,n_y\right)=\frac{2\exp\left[-j\left|\xi\right|\sqrt{n_2^2-n_y^2}\pm jm\arccos\left(n_y/n_2\right)\right]}{\sqrt{n_2^2-n_y^2}}$$

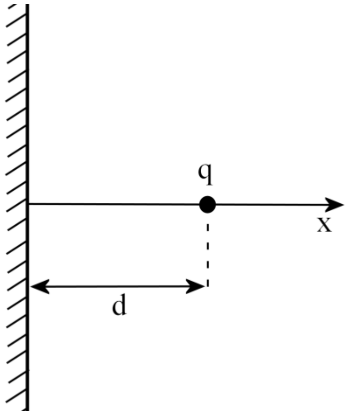
$$R_{21}\left(n_y\right)=\begin{cases}\dfrac{n_{1x}-n_{2x}}{n_{1x}+n_{2x}} & \text{E-polarization}\\[2ex] \dfrac{\varepsilon_2 n_{1x}-\varepsilon_1 n_{2x}}{\varepsilon_2 n_{1x}+\varepsilon_1 n_{2x}} & \text{H-polarization}\end{cases}$$

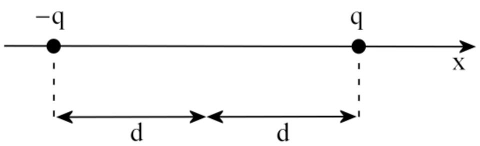
where: $n_x=\sqrt{n^2-n_y^2}$

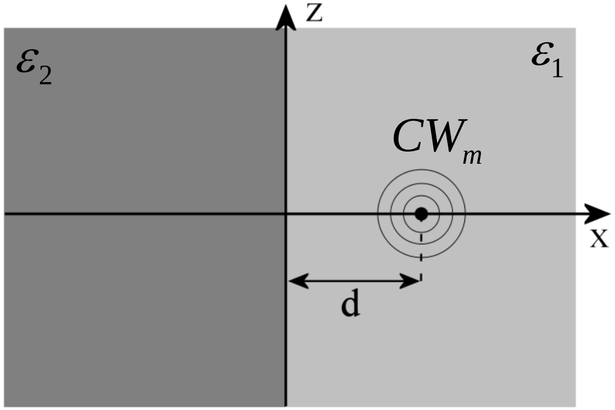
We find extremely complicated expressions!

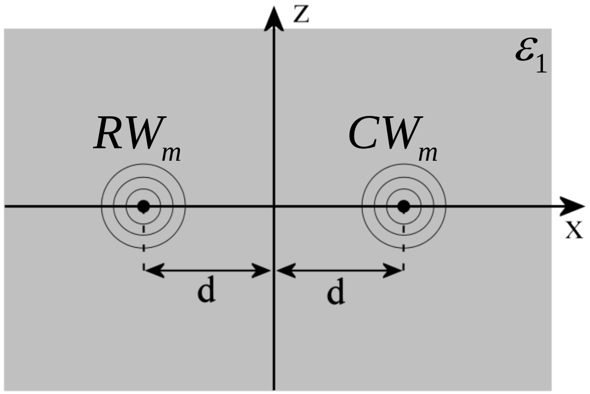
Image principle

Try to think different!









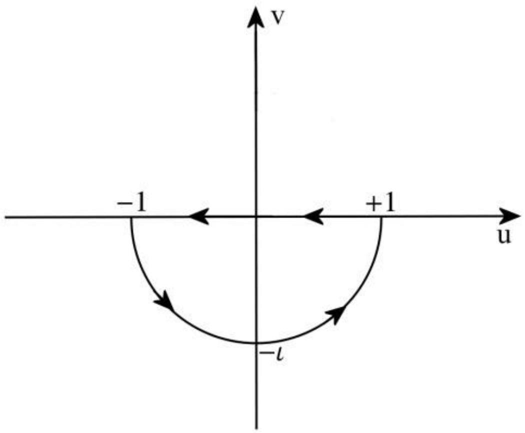
Fresnel coefficients

$$R_{21}(n_y)=\sum_{k=0}^{+\infty}\gamma_k n_y^k \quad ?$$

It is not possible R_{21} has two polar singularities in the complex plane.

Conformal mapping

$$w=n_y\pm\sqrt{n_y^2-1} \Rightarrow n_y=\frac{w^2+1}{2w}$$



$$R_{21}(n_y)=\sum_{k=0}^{+\infty}\gamma_k w^k \quad \Rightarrow \quad R_{21}(n_y)=\sum_{k=0}^{+\infty}\gamma_{2k}e^{-2j\arccos|n_y|}$$

Fresnel coefficients

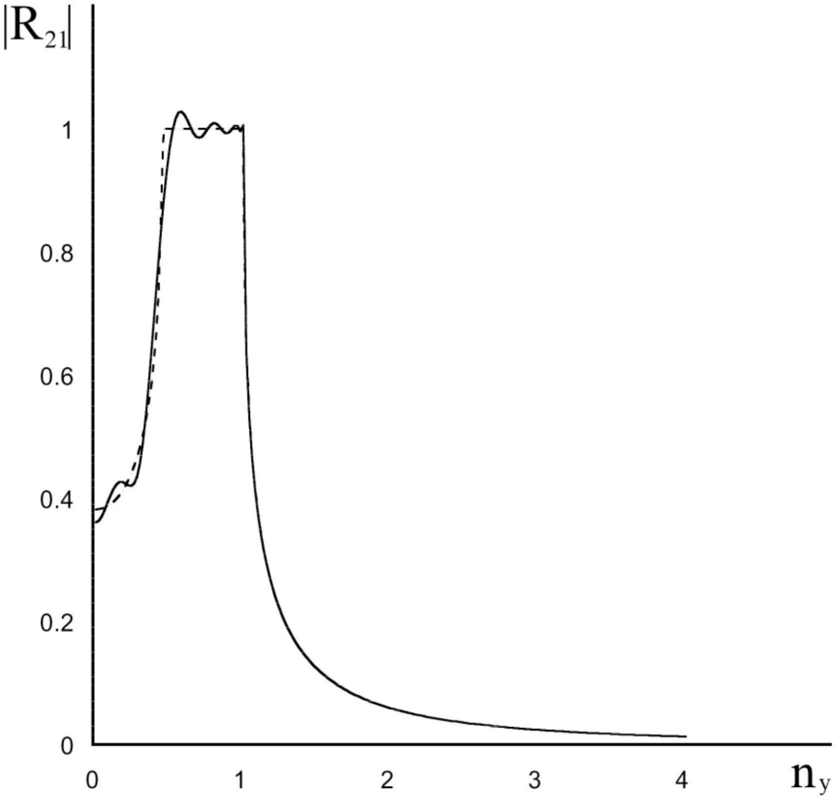
$$R_{21}\left(n_y\right)=\sum_{k=0}^{+\infty}\gamma_{2k}e^{-2j\arccos\left|n_y\right|}$$

$$\Downarrow$$

$$RW_m\left(\xi,\zeta\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{21}\left(n_y\right)F_m\left(\xi,n_y\right)\exp\left(-jn_y\zeta\right)\mathrm{d}n_y$$

$$\Downarrow$$

Because the absolute value,
we must split the integral into two!



Solid line: series truncated at the 10th element

The analytical solution

Let us define the following integral:

$$I_m\left(\xi,n_y\right)=\frac{1}{2\pi}\int_0^{+\infty}F_m\left(\xi,n_y\right)\exp[-\mathrm{j}\zeta\,\mathrm{n}_y]\,\mathrm{dn}_y$$

Then, inserting: $R_{21}\left(n_y\right)=\sum_{k=0}^{+\infty}\gamma_{2k}e^{-2j\arccos\left|n_y\right|}$

Into: $RW_m\left(\xi,\zeta\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}R_{21}\left(n_y\right)F_m\left(\xi,n_y\right)\exp\left(-jn_y\zeta\right)\mathrm{d}n_y$

We obtain:

$$RW_m\left(\xi,\zeta\right)=\sum_{k=0}^{+\infty}\gamma_{2k}\left[I_{m-2k}\left(\xi,\zeta\right)+\left(-1\right)^m I_{-m-2k}\left(\xi,-\zeta\right)\right]$$

The analytical solution

Let us define the following integral:

This integral can be solved analytically!

$$I_m(\xi, n_y) = \frac{1}{2\pi} \int_0^{+\infty} F_m(\xi, n_y) \exp[-\mathrm{j}\zeta\, \mathrm{n}_y] \mathrm{dn}_y$$

Then, inserting: $$R_{21}(n_y) = \sum_{k=0}^{+\infty} \gamma_{2k} e^{-2j \arccos|n_y|}$$

Into: $$RW_m(\xi, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{21}(n_y) F_m(\xi, n_y) \exp(-jn_y\zeta) \mathrm{d}n_y$$

We obtain:

$$RW_m(\xi, \zeta) = \sum_{k=0}^{+\infty} \gamma_{2k} \left[I_{m-2k}(\xi, \zeta) + (-1)^m I_{-m-2k}(\xi, -\zeta) \right]$$

The analytical solution (2)

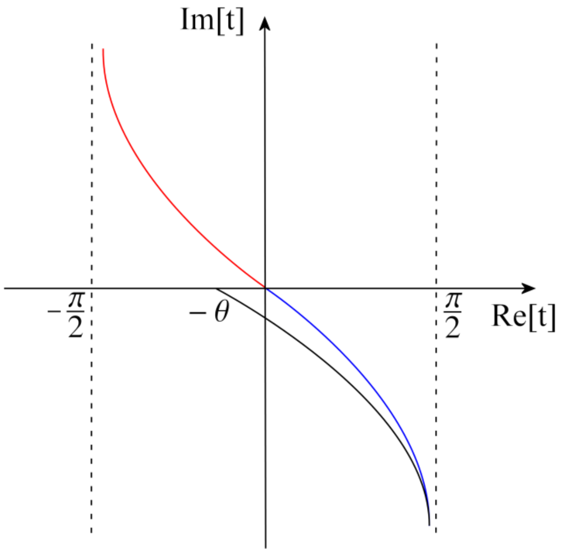
$$I_m(\xi, n_y) = \frac{1}{2\pi} \int_0^{+\infty} F_m(\xi, n_y) \exp[-\mathrm{j}\zeta\, \mathrm{n}_y] \mathrm{dn}_y$$

This integral is closely correlated with the Sommerfeld integral

Posing $n_y = \sin t$

$$I_m(\xi, n_y) = \frac{j^m}{\pi} \int_{-\theta}^{\frac{\pi}{2} - j\infty} \exp[-\mathrm{j}\rho \cos t - \mathrm{j} m(t+\theta)] \mathrm{d}t$$

Now we can solve the integral on the real and on the imaginary axis separately



The analytical solution (2)

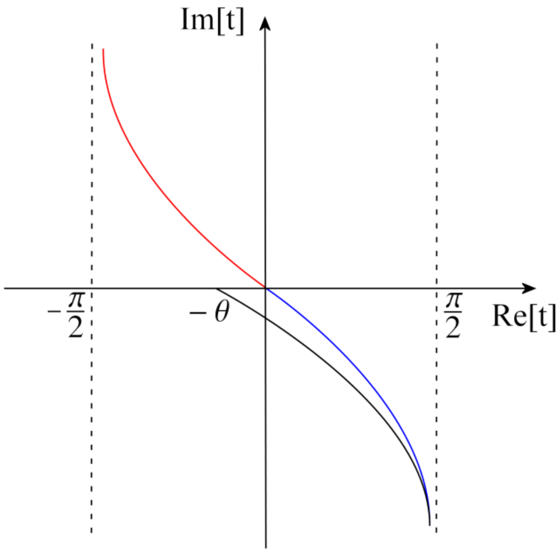
$$I_m\left(\xi, n_y\right) = \frac{1}{2\pi}\int_0^{+\infty} F_m\left(\xi, n_y\right)\exp[-\mathrm{j}\zeta\,\mathrm{n}_y]\,\mathrm{dn}_y$$

This integral is closely correlated with the Sommerfeld integral

Posing $n_y = \sin t$

t is complex!

$$I_m\left(\xi, n_y\right) = \frac{j^m}{\pi}\int_{-\theta}^{\frac{\pi}{2}-j\infty} \exp[-\mathrm{j}\rho\cos t - \mathrm{j}m\left(t+\theta\right)]\,\mathrm{d}t$$



Now we can solve the integral on the real and on the imaginary axis separately

The analytical solution (3)

On the real axis:

$$\frac{j^m}{\pi}\int_{-\theta}^{\frac{\pi}{2}} \exp[-\mathrm{j}\rho\cos t - \mathrm{j}m(t+\theta)]\,\mathrm{d}t = \sum_{p=-\infty}^{+\infty} a_{m,p} J_p(\rho)$$

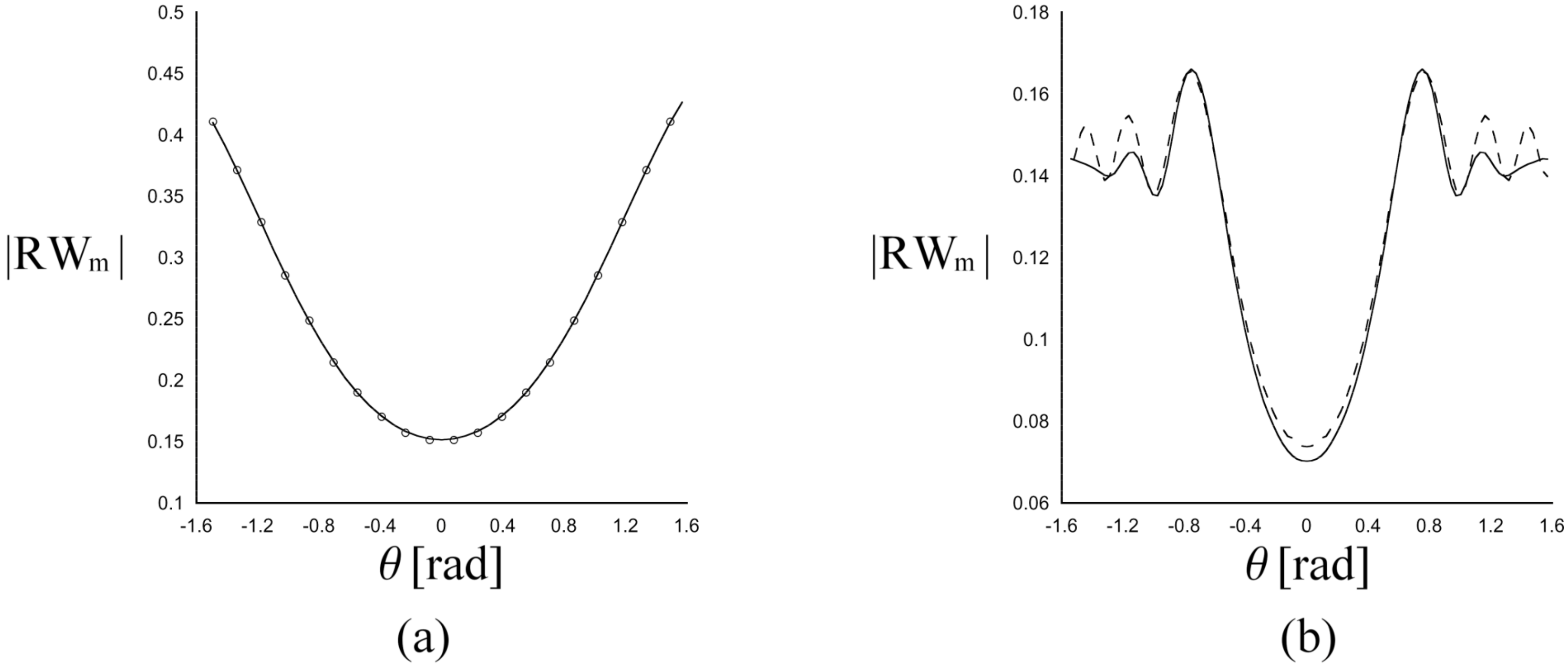
with: $$a_{m,p} = \frac{2}{\pi} j^{m-p}\left(\frac{\pi}{4}+\frac{\theta}{2}\right) e^{-j(m+p)\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} \sinh\left[(m+p)\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

On the imaginary axis:

$$\frac{1}{\pi}\int_{0}^{+\infty} \exp[-\rho\sinh v - mv]\,\mathrm{d}v = A_m(\rho)$$

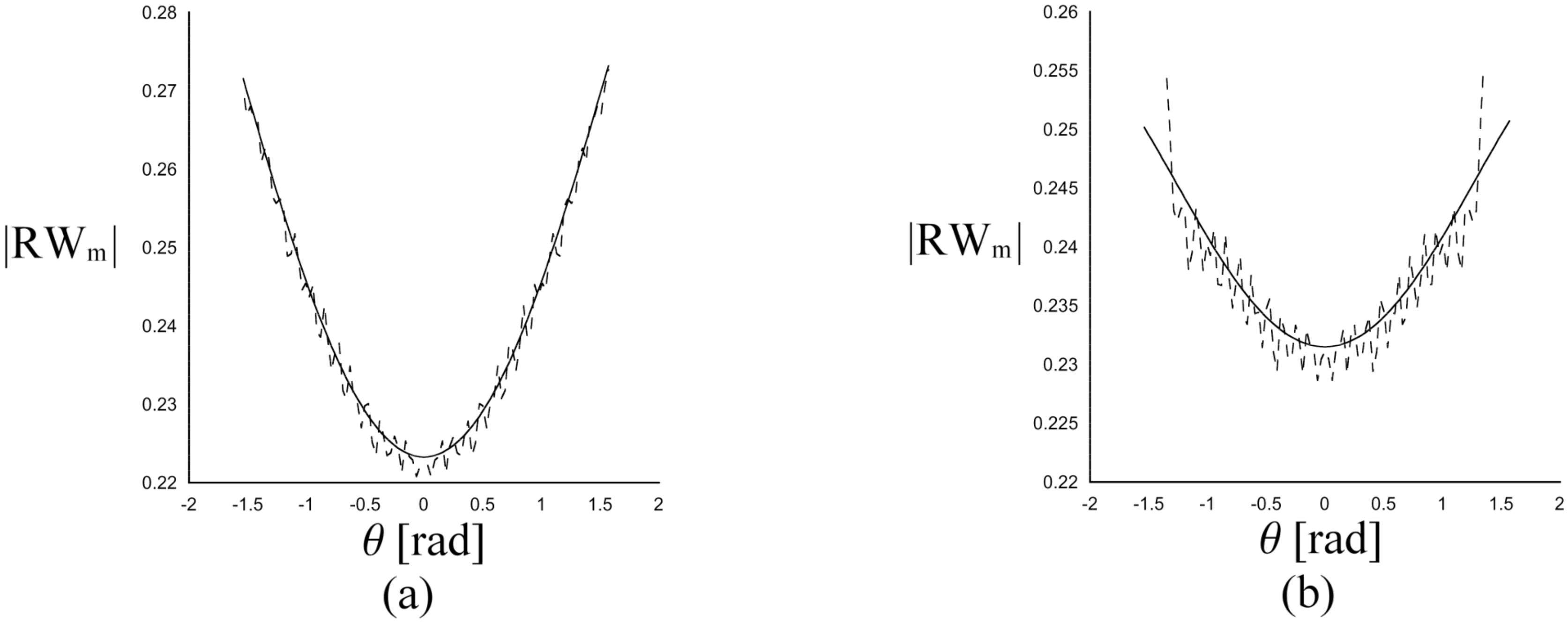
Where $A_m(\rho)$ is the Associate Anger-Weber function, solution of the inhomogeneous Bessel differential equation

Numerical results



Amplitude of the RW of order 0, when medium 1 is a vacuum and medium 2 has relative permittivity 10. The RW is computed in $\rho = 5$ (a), and in $\rho = 30$ (b). Analytical method (solid line). Numerical method (circles (a), dashed line (b)).

Numerical results (2)



Amplitude of the RW of order 0, when medium 1 is a vacuum and medium 2 has relative permittivity 10. The RW is computed in ρ = 0.6 (a), and in ρ = 0.3 (b). Analytical method (solid line). Numerical method (dashed line).

Conclusions

- The Cylindrical Wave Approach has been presented in the case of lossless media
- The convergence problems arise in lossy media has been presented
- A first numerical solution to the convergence problem has been proposed
- Limitations of the numerical solutions have been pointed out
- A power expansion of the Fresnel reflection coefficients have been presented
- The analytical solution, valid in the whole complex plane, to the reflected cylindrical wave has been derived